113 Class Problems: Polynomial Fazorization

1. (a) The polynomial $x^2 + 1$ has no roots in $\mathbb{R}$. However when we go to $\mathbb{C}$ two roots appear, namely $\pm i$. Why is this nowhere near enough to conclude $\mathbb{C}$ is algebraically closed?

   (b) Prove that the quotient ring $\mathbb{R}[x]/(x^2 + 1)$ is a field. What familiar field is it isomorphic to?

   Solutions:

a) Just because x^2+1 has a root in $\mathbb{C}$, it does not imply that every non-constant polynomial does

b) $\phi : \mathbb{R}[x] \longrightarrow \mathbb{C}$ is a surjective homomorphism
$f(x) \longrightarrow f(i)$

Claim $\ker\phi = (x^2+1)$. $\left(\Rightarrow \mathbb{R}[x]/(x^2+1) \cong \mathbb{C}\right)$

$i^2+1=0 \Rightarrow (x^2+1) \subset \ker\phi$. Assume $\exists f(x) \in \ker\phi$

$x^2+1 \nmid f(x) \Rightarrow f(x) = g(x)\, x^2+1 + r(x)$, $r(x) \neq 0$, $\deg(r(x)) < 2$

$\Rightarrow r(i) = 0 \Rightarrow i \in \mathbb{R}$. Contradiction. Hence $\ker\phi = (x^2+1)$

2. Let F be a field and $f(x) \in F[x]$ be an irreducible polynomial of degree $n > 1$. Does there exist $\alpha \in F$ such that $f(\alpha) = 0_F$?

   Solutions:

$f(x) \in F[x]$ irreducible, $\deg(f(x)) > 1$

Claim $f(x)$ admits no roots in F.

Proof Let $\alpha \in F$ such that $f(\alpha) = 0$

$\Rightarrow f(x) = (x-\alpha)\, g(x)$, for some $g(x) \in F[x]$

$\deg(f(x)) > 1 \Rightarrow \deg(g(x)) > 1 \Rightarrow (x-\alpha), g(x) \notin F[x]^*$

$\Rightarrow f(x)$ reducible. Contradiction

$\square$

3. Is it possible for a finite field F to be algebraically closed?

Solutions:

No. Let $F = \{a_1, \dots, a_n\}$ then

$f(x) = (x - a_1)(x - a_2) \dots (x - a_n) + 1_F$ is degree $n > 1$

admits no roots in F.

4. (a) Recall that $\mathbb{Q}[i] = \{a + bi | a, b \in \mathbb{Q}\}$ is a field. Is it algebraically closed?

   (b) Is the field $\mathbb{C}(z)$ algebraically closed?

Solutions:

a)

No. $\sqrt{2} \notin \mathbb{Q} \Rightarrow \sqrt{2} \notin \mathbb{Q}[i] \Rightarrow x^2 - 2 \in \mathbb{Q}[i][x]$

has no roots in $\mathbb{Q}[i]$

b) No

Observe that $\nexists\, f(z), g(z) \in \mathbb{C}[z]$ such that

$$\left(\frac{f(z)}{g(z)}\right)^2 = z$$

$\Rightarrow x^2 - z \in \mathbb{C}(z)[x]$ has no root in $\mathbb{C}(z)$